Nov 1 –
Week 9

Lecture

Line Integrals

(I) Parametric curves
(II) Line integral of functions
(III) Curves and their parametrization
(IV) Vector fields and the line integral of v.f's.
(V) flow and flux

(I) Parametric Curves

A map $\vec{r}$ from an interval I to $\mathbb{R}^2$, or $\mathbb{R}^3$ is called a parametric curve if $\forall t \in I$,

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}, \text{ or}$$
$$= x(t)\,\hat{i} + y(t)\,\hat{j}$$

the functions $x(t)$, $y(t)$, $z(t)$ are continuous. Usually take $I = [a,b]$, it is smooth, ie, C^1 if

$$x'(t),\ y'(t),\ z'(t) \text{ exist and are continuous.}$$

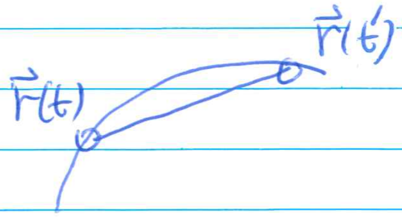
It is regular if $|\vec{r}\,'(t)| = \left(x'^2(t) + y'^2(t) + z'^2(t)\right)^{1/2} > 0, \forall t.$

Not always stated explicitly, most p-curves in our study are smooth and regular.

Some concept associated to p-curve.

Imagine $\vec{r}(t)$ is the trajectory of an object where t is the time and $\vec{r}(t)$ the position of the object at time t.

The distance traveled from $\vec{r}(t)$ to $\vec{r}(t')$ is approx. given by



$$|\vec{r}(t')-\vec{r}(t)|$$

$$= \left[(x(t')-x(t))^2+(y(t')-y(t))^2\right]^{1/2} \quad (n=2)$$

(mean-value thm) $$= \left[x'^2(t^*)(\Delta t)^2 + y'^2(t^{**})(\Delta t)^2\right]^{1/2}, \quad t^*, t^{**} \in [t, t']$$

$\therefore$ approximate speed

$$= \frac{1}{\Delta t}|\vec{r}(t')-\vec{r}(t)|$$

$$= \left[x'^2(t^*)+y'^2(t^*)\right]^{1/2}$$

Letting $t' \to t$, the speed of the object at t is

$$|\vec{r}'(t)| = \sqrt{x'^2(t)+y'^2(t)} \quad (n=2)$$

$$= \sqrt{x'^2(t)+y'^2(t)+z'^2(t)} \quad (n=3)$$

The vector $\vec{r}'(t)$ is called the velocity of the object at t.

The unit vector

$$\vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

is the tangent vector of the p-curve at $\vec{r}(t)$.

(II) Line integral of functions

Let $\vec{r}: [a,b] \to \mathbb{R}^2, \mathbb{R}^3$ be a p-curve and f is continuous function defined on the image of $\vec{r}$, i.e.,

$\vec{r}([a,b])$, which is a subset of $\mathbb{R}^2, \mathbb{R}^3$. Imagine $f \geq 0$ is a density on the "curve" $\vec{r}([a,b])$. The approximate mass is

$$\sum_j f(\vec{r}(t_j^*))\,|\vec{r}(t_{j+1}) - \vec{r}(t_j)| \qquad t_j^* \text{ tag pt}$$

where t_j's is a partition on $[a,b]$. Using

$$|\vec{r}(t_{j+1}) - \vec{r}(t_j)|$$

$$= \left(x'^2(t_j^1) + y'^2(t_j^2)\right)^{\frac{1}{2}} \Delta t_j, \quad \text{(see above)},$$

$$t_j^1, t_j^2 \in [t_j, t_{j+1}],$$

$$\sum_j f(\vec{r}(t_j^*))\sqrt{x'^2(t_j^1) + y'^2(t_j^2)}\,\Delta t_j$$

$$\to \int_a^b f(\vec{r}(t))\,|\vec{r}'(t)|\,dt. \quad \text{as all } \Delta t_j \to 0$$

So, we define the line integral of f along the p-curve $\vec{r}$ by

$$\underbrace{\int_{\vec{r}} f\,ds}_{\text{notation}} = \int_a^b f(\vec{r}(t))\,|\vec{r}'(t)|\,dt.$$

From the deviation, we know

- $f \geq 0$, $\int_{\vec{r}} f\,ds$ gives the mass of $\vec{r}([a,b])$ with density f
- $f \equiv 1$, $\int_{\vec{r}} ds$ gives the length of $\vec{r}([a,b])$.

e.g 1. Evaluate

$$\int_{\vec{r}} (2xy+\sqrt{z})\,ds \quad \text{where } \vec{r} : [0,\pi] \to \mathbb{R}^3 \text{ given by}$$

$$\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}.$$

$$f(\vec{r}(t)) = 2\cos t \sin t + \sqrt{t}$$

$$= \sin 2t + \sqrt{t}.$$

$$\vec{r}\,'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}$$

$$|\vec{r}\,'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}$$

$$\therefore \int_{\vec{r}} (2xy+\sqrt{z})\,ds = \int_0^{\pi} (\sin 2t + \sqrt{t})\sqrt{2}\,dt$$

$$= \sqrt{2}\left(-\frac{\cos 2t}{2} + \frac{2}{3}t^{\frac{3}{2}}\right)\Bigg|_0^{\pi}$$

$$= \frac{2\sqrt{2}}{3}\pi^{3/2}. \quad \#$$

III Curves and their Parametrization

A curve is a subset of $\mathbb{R}^2$, $\mathbb{R}^3$ that looks like a "curve". Examples are

circle : $(x-x_0)^2 + (y-y_0)^2 = r^2$,

line segments between 2 pts,

ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

parabola : $y = ax^2 + b$, $a \neq 0$.

A mathematical definition is: A subset C in $\mathbb{R}^2, \mathbb{R}^3$ is a curve if there exists a regular, smooth parametric curve $\vec{r}: [a,b] \to \mathbb{R}^2, \mathbb{R}^3$ maps 1-1 onto this subset, in other words,

$$C = \vec{r}([a,b]).$$

This definition applies to the situation where C has different endpoints. When C is a closed curve, we require $\vec{r}$ maps $[a,b)$ 1-1 onto C and $\vec{r}(a) = \vec{r}(b)$.

Here are some standard parametrization of curves.

~ the circle $(x-x_0)^2 + (y-y_0)^2 = r^2$

$$\vec{r}(t) = (r\cos t + x_0)\hat{i} + (r\sin t + y_0)\hat{j}, \quad t \in [0, 2\pi].$$

~ Line segment connecting $\vec{P}$ and $\vec{Q}$:

$$\vec{r}(t) = \vec{P} + t(\vec{Q} - \vec{P}), \qquad t \in [0,1]$$

$\vec{r}(0) = \vec{P}$, $\vec{r}(1) = \vec{Q}$. As t increases from 0 to 1, $\vec{r}(t)$ runs from $\vec{P}$ to $\vec{Q}$ along the line in constant speed.

~ Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$x(t)\hat{i} + y(t)\hat{j} = a\cos t\,\hat{i} + b\sin t\,\hat{j}, \quad t \in [0,1].$$

~ parabola $y = ax^2 + b$, $a \neq 0$

$$\vec{r}(x) = x\hat{i} + (ax^2 + b)\hat{j}, \qquad x \in (-\infty, \infty)$$

using x as the t parameter.

~ when the curve is the graph of $f(x)$ over $[a,b]$,

$$\vec{r}(x) = x\hat{i} + f(x)\hat{j}, \quad x \in [a,b].$$

Let f be a continuous function on a curve C. We define the <u>line integral of f along C</u> to be

$$\int_C f\,ds = \int_{\vec{r}} f\,ds$$

$$= \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt, \text{ where } \vec{r} \text{ is a}$$

parametrization of C. From the derivation of $\int_{\vec{r}} f\,ds$ it is clear that $\int_{\vec{r}} f\,ds$ only depends on C (we will verify this in a supplementary exercise.)

e.g. 2 Find L be the line segment connecting $(1,0,1)$ and $(1,-2,3)$. Find its length ~~& its~~ line integral.

Choose the parametrization

$$\vec{l}(t) = (1,0,1) + t\big((1,-2,3)-(1,0,1)\big)$$

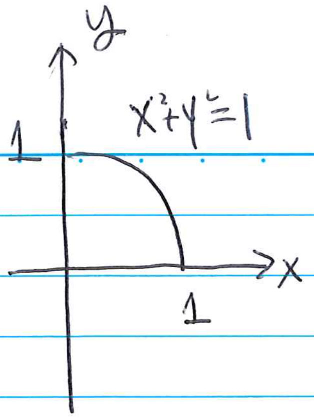
$$= (1,0,1) + t(0,-2,2), \quad t \in [0,1]$$

$$\vec{l}\,'(t) = (0,-2,2), \quad |\vec{l}\,'(t)| = 2\sqrt{2}.$$

$$\therefore \text{length of } L = \int_0^1 1\cdot 2\sqrt{2}\,dt = 2\sqrt{2} \ \#$$

e.g. 3 Let C be the arc of $x^2+y^2=1$ in 1st quadrant. Find

$$\int_C x\,ds.$$



We will carry out the calculation using three different parametrizations.

① $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}, \quad t \in [0, \frac{\pi}{2}]$

$\vec{r}'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j}, \quad |\vec{r}'(t)| = 1.$

$\therefore \int_C x\,ds = \int_0^{\pi/2} \cos t \times 1\,dt = \sin t\Big|_0^{\pi/2} = 1.$

② $\vec{r}_2(t) = \cos t^2\,\hat{i} + \sin t^2\,\hat{j}, \quad t \in [0, \sqrt{\frac{\pi}{2}}].$

$\vec{r}_2'(t) = -2t\sin t^2\,\hat{i} + 2t\cos t^2\,\hat{j}$

$|\vec{r}_2'(t)| = (4t^2\sin^2 t^2 + 4t^2\cos^2 t^2)^{1/2} = 2t.$

$\int_C x\,ds = \int_0^{\sqrt{\frac{\pi}{2}}} \cos t^2\, 2t\,dt = \int_0^{\pi/2} \cos z\,dz = 1.$

③ $\vec{r}_3(x) = x\,\hat{i} + \sqrt{1-x^2}\,\hat{j}, \quad x \in [0,1]$

$\vec{r}_3'(x) = \hat{i} + \frac{-x}{\sqrt{1-x^2}}\,\hat{j}$

$|\vec{r}_3'(x)| = \sqrt{1 + \left(\frac{-x}{\sqrt{1-x^2}}\right)^2} = \frac{1}{\sqrt{1-x^2}}.$

$\therefore \int_C x\,ds = \int_0^1 \frac{x\,dx}{\sqrt{1-x^2}} = \frac{1}{2}\int_0^1 \frac{dz}{\sqrt{1-z}} = 1.$

So, no matter which parametrization you choose the end result is the same.

The following theorem shows all parametrizations of a curve can be classified into 2 classes.

Theorem 1 Let C be a curve with different endpoints $\vec{P}$ and $\vec{Q}$.

ⓐ Let $\vec{r}_1 : [a,b] \to C$, $\vec{r}_2 : [\alpha, \beta] \to C$ be 2 parametriz. of C satisfying $\vec{r}_1(a) = \vec{r}_2(\alpha) = \vec{P}$, $\vec{r}_1(b) = \vec{r}_2(\beta) = \vec{Q}$. There exists $\varphi : [a,b] \underset{\text{onto}}{\overset{1-1}{\to}} [\alpha, \beta]$, $\varphi(a) = \alpha$, $\varphi(b) = \beta$, $\varphi' > 0$, s.t.

$$\vec{r}_2(\varphi(t)) = \vec{r}_1(t), \quad \forall t \in [a, b].$$

ⓑ Let $\vec{r}_1 : [a,b] \to C$, $\vec{r}_2 : [\alpha, \beta] \to C$ be 2 paramet... of C satisfying $\vec{r}_1(a) = \vec{r}_2(\beta) = \vec{P}$, $\vec{r}_1(b) = \vec{r}_2(\alpha) = \vec{Q}$. There exists $\varphi : [a,b] \underset{\text{onto}}{\overset{1-1}{\to}} [\alpha, \beta]$, $\varphi(a) = \beta$, $\varphi(b) = \alpha$, $\varphi' < 0$ s.t.

$$\vec{r}_2(\varphi(t)) = \vec{r}_1(t).$$

Sketch of Proof. ⓐ $\forall t$, $\vec{r}_1(t)$ is a point on C, since $\vec{r}_2$ maps $[\alpha, \beta]$ 1-1 onto C, there is a unique $z \in [\alpha, \beta]$ such that $\vec{r}_2(z) = \vec{r}_1(t)$. The correspondence $t \mapsto z$ set up a map from $[a,b]$ 1-1 onto $[\alpha, \beta]$ s.t. $\vec{r}_2(\varphi(t)) = \vec{r}_1(t)$. By differentiating

$$\vec{r}_2{}'(\varphi(t))\, \varphi'(t) = \vec{r}_1{}'(t) \qquad (\text{Chain rule})$$

$$|\vec{r}_2{}'(\varphi(t))|\,|\varphi'(t)| = |\vec{r}_1{}'(t)|$$

Since $|\vec{r}_2{}'(z)| > 0$, $|\vec{r}_1{}'(t)| > 0$, we have $|\varphi'(t)| > 0$. As $\varphi(a) = \alpha$, $\varphi(b) = \beta \Rightarrow \varphi'(t) \geq 0$. We conclude $\varphi'(t) > 0$.

(b) Can be proved similarly.

Note When C is a closed curve, the same conclusion holds. That's, given two parametrizations of C, there exists $\varphi : [a,b] \xrightarrow[\text{onto}]{1\text{-}1} [\alpha,\beta]$ such that $\vec{r}_2(\varphi(t)) = \vec{r}_1(t)$. Either

$\varphi' > 0$ or $\varphi' < 0$.

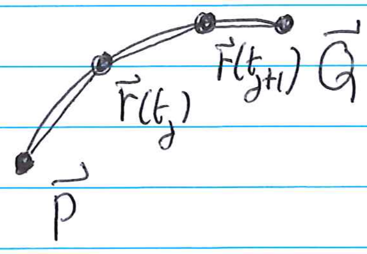
(IV) Vector fields and the line integrals of v.f's.

A vector field $\vec{F}(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$ in $\mathbb{R}^3$ and $\vec{F}(x,y) = M(x,y)\hat{i} + N(x,y)\hat{j}$ in $\mathbb{R}^2$.

Suppose an object is moving along a p-curve $\vec{r} = [a,b] \to \mathbb{R}^2, \mathbb{R}^3$ from $\vec{P} = \vec{r}(a)$ to $\vec{Q} = \vec{r}(b)$. We want to introduce a definition of work done a force field $\vec{F}$ acting on the object.

For simplicity, take $n=2$ case. Recall that under a constant force field $\vec{F}$, the work done from point $\vec{P}$ to $\vec{Q}$ is given by

$$\vec{F}\cdot(\vec{Q}-\vec{P}).$$



Now, when the object moves from $\vec{P}$ to $\vec{Q}$ along C. Introduce a partition on $[a,b]$ so that C is approximated by the sum of line segments connecting $\vec{r}(t_j)$ to $\vec{r}(t_{j+1})$.

On the line segment $\vec{r}(t_j)$ to $\vec{r}(t_{j+1})$, the force is approximately a constant given by

$$\vec{F}(\vec{r}(t_j^*)), \quad t_j^* \in [t_j, t_{j+1}]$$

$\therefore$ Approximate work done

$$= \sum \vec{F}(\vec{r}(t_j^*)) \cdot (\vec{r}(t_{j+1}) - \vec{r}(t_j))$$

$$= \sum \vec{F}(\vec{r}(t_j^*)) \cdot \vec{r}\,'(t_j^{**}) \Delta t_j, \quad t_j^{**} \in [t_j, t_{j+1}],$$

$\Delta t_j \to 0$

$$\to \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt.$$

We define the line integral of $\vec{F}$ along the p-curve $\vec{r}$ to be

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt.$$

When $\vec{F}$ is a force field, this integral gives the work done of the force field on an object moving from $\vec{r}(a)$ to $\vec{r}(b)$.

e.g. 4. Find

$$\int_{\vec{r}} \vec{F} \cdot d\vec{r}$$

when $\vec{F} = z\hat{i} + xy\hat{j} - y^2\hat{k}$

$$\vec{r}(t) = t^2\hat{i} + t\hat{j} + \sqrt{t}\,\hat{k}, \quad t \in [0,1].$$

$$\vec{r}\,'(t) = 2t\hat{i} + \hat{j} + \frac{1}{2}t^{-\frac{1}{2}}\hat{k}$$

$$\vec{F}(\vec{r}(t)) = \sqrt{t}\,\hat{i} + t^3\hat{j} - t^2\hat{k}$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_0^1 \left(\sqrt{t}\,\hat{i} + t^3\hat{j} - t^2\hat{k}\right)\left(2t\,\hat{i} + \hat{j} + \frac{1}{2}\frac{1}{\sqrt{t}}\,\hat{k}\right) dt$$

$$= \int_0^1 \left(2t^{3/2} + t^3 - \frac{1}{2}t^{3/2}\right) dt$$

$$= \frac{17}{20} \quad \#$$

Different notation for the line integral of v.f's.

Let $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$

$r'(t) = x'(t)\,\hat{i} + y'(t)\,\hat{j} + z'(t)\,\hat{k}$

$$\int_C \vec{F}\cdot d\vec{r} \stackrel{\text{def}}{=} \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

$$= \int_a^b \left(M(\vec{r}(t))\,x'(t) + N(\vec{r}(t))\,y'(t) + P(\vec{r}(t))\,z'(t)\right) dt$$

Suggest,

$\int_C \vec{F}\cdot d\vec{r}$ to express as

$$\int_C M\,dx + N\,dy + P\,dz.$$

Invariance of the line integral of v.f in parametrizat.

Theorem 2 Let $\vec{r}_1$ and $\vec{r}_2$ be 2 parametrizations of C in the same direction, then

$$\int_{\vec{r}_1} \vec{F}\cdot d\vec{r} = \int_{\vec{r}_2} \vec{F}\cdot d\vec{r}.$$

when they are in opposite direction,

$$\int_{\vec{r}_1} \vec{F}\cdot d\vec{r} = -\int_{\vec{r}_2} \vec{F}\cdot d\vec{r}.$$

Pf. Let $\vec{r}_1$ on $[a,b]$ and $\vec{r}_2$ on $[\alpha,\beta]$ both the same direction. By Theorem 1, $\exists\ \varphi : [a,b] \to [\alpha,\beta]$, $\varphi(a)=\alpha$, $\varphi(b)=\beta$, $\varphi'>0$, s.t. $\vec{r}_2(\varphi(t)) = \vec{r}_1(t)$. Now,

$$\int_{\vec{r}_2} \vec{F}\cdot d\vec{r} = \int_\alpha^\beta \vec{F}(\vec{r}_2(z))\cdot \vec{r}_2{}'(z)\,dz$$

$\vec{r}_2{}'(z)\varphi'(t) = \vec{r}_1{}'(t)$ by Chain Rule

$$= \int_{\varphi(a)}^{\varphi(b)} \vec{F}(\vec{r}_2(z))\cdot \frac{\vec{r}_1{}'(t)}{\varphi'(t)}\,dz$$

Change of variables formula

$$= \int_a^b \vec{F}(\vec{r}_1(t))\,\frac{\vec{r}_1{}'(t)}{\varphi'(t)}\,\varphi'(t)\,dt$$

$$= \int_a^b \vec{F}(\vec{r}_1(t))\,\vec{r}_1{}'(t)\,dt$$

$$= \int_{\vec{r}_1} \vec{F}\cdot d\vec{r}.$$

When $\vec{r}_1$ and $\vec{r}_2$ are in opposite direction, $\varphi(a)=\beta$, $\varphi(b)=\alpha$ and $\varphi'<0$. As above

$$\int_{\vec{r}_2} \vec{F}\cdot d\vec{r} = \int_{\varphi(a)}^{\varphi(b)} \vec{F}(\vec{r}_2(z))\cdot\frac{\vec{r}_1{}'(t)}{\varphi'(t)}\,dz$$

$$= \int_b^a \vec{F}(\vec{r}_1(t))\cdot\frac{\vec{r}_1(t)}{\varphi'(t)}\,\varphi'(t)\,dt$$

$$= \int_b^a \vec{F}(\vec{r}_1(t))\cdot\vec{r}_1{}'(t)\,dt = -\int_{\vec{r}_1} \vec{F}\cdot d\vec{r}. \quad \#$$

(V) Flows and Flux.

Let C be a curve with a direction and $\vec{F}$ a v.f. on C.

We interpret $\vec{F}$ as the velocity of some fluid at a fixed time. Then

$$\vec{F}\cdot\vec{T}$$

is the amount of the fluid passing the curve in unit time. Hence,

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r}$$

can be understood as the amount of fluid passing along C in unit time, call it <u>the flow of $\vec{F}$ along C</u>.

When C is closed, call it the <u>circulation of $\vec{F}$ around C</u>,

e.g. 5 Find the circulation of $\vec{F}=(x-y)\hat{i}+x\hat{j}$ around the circle $\vec{r}(t)=\cos t\,\hat{i}+\sin t\,\hat{j}$, $t\in[0,2\pi]$.

$$\vec{F}(\vec{r}(t)) = (\cos t-\sin t)\hat{i}+\cos t\,\hat{j}$$
$$\vec{r}\,'(t) = -\sin t\,\hat{i}+\cos t\,\hat{j}$$

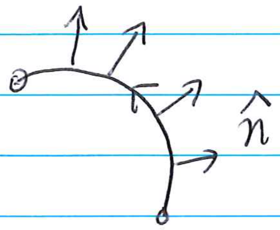
$$\therefore \text{circulation} = \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$$= \int_0^{2\pi} [(\cos t-\sin t)(-\sin t)+\cos t\cos t]\,dt$$

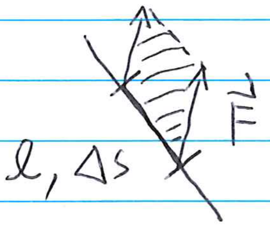
$$= \int_0^{2\pi} (-\sin t\cos t+1)\,dt$$

$$= 2\pi \quad \#$$

Let C be a curve in $\mathbb{R}^2$ with a choice of normal $\hat{n}$.



Let $\vec{F}$ be the velocity v.f. for some fluid on C



The amount of fluid passing through the part l in Δt is given by the area of the //-gram, ie

$$\Delta s \times \underbrace{\vec{F}\cdot\hat{n}}_{\text{height}} \Delta t$$

Taking limit and integrate along C, it is

$$\left(\int_C \vec{F}\cdot\hat{n}\, ds\right)\Delta t.$$

Hence, the amount of fluid with velocity $\vec{F}$ passing through C in unit time; that is, the <u>flux</u> is,

$$\int_C \vec{F}\cdot\hat{n}\, ds$$

When C is a closed curve, $\hat{n}$ will be taken to be the unit outer normal. Let $r(t)$ be a parametrization in anticlockwise direction, then $r'(t) = x'(t)\,\hat{i} + y'(t)\,\hat{j}$ points to the tangential direction, so

$$y'(t)\,\hat{i} - x'(t)\,\hat{j}$$

points to the normal direction. One can check that the outer normal is

$$\hat{n} = \frac{y'\hat{i} - x'\hat{j}}{\sqrt{x'^2 + y'^2}}.$$

When $\vec{F} = M\hat{i} + N\hat{j}$, the flux is

$$\oint_C \vec{F}\cdot\hat{n}\, ds = \oint_C (M\hat{i} + N\hat{j})(y'\hat{i} - x'\hat{j})\, dt$$

$$= \oint_C My' - Nx'\, dt$$

$$= \oint_C M\,dy - N\,dx.$$

e.g. 6 Find the flux of $\vec{F} = (x-y)\hat{i} + x\hat{j}$ across the circle $x^2+y^2=1$.

Take $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$

$\vec{r}'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} = x'\hat{i} + y'\hat{j}$

$$\text{flux} = \oint_C M\,dy - N\,dx$$

$$= \int_0^{2\pi} [(\cos t - \sin t)\cos t - \cos t(-\sin t)]\,dt$$

$$= \int_0^{2\pi} \cos^2 t\, dt$$

$$= \pi \quad \#$$